7.45 ⋆⋆ **(a)** If you look at the definition (7.95) of A_{jk}, you will see that A_{jk} differs from A_{kj} only in the order of the two terms in the scalar product. Since the scalar product is commutative, these two expressions are equal.

(b) If we consider first the case of just two variables, the sum in question is

$$\begin{aligned} S = \sum_{j,k} A_{jk} v_j v_k &= A_{11} v_1^2 + A_{12} v_1 v_2 + A_{21} v_2 v_1 + A_{22} v_2^2 \\ &= A_{11} v_1^2 + 2A_{12} v_1 v_2 + A_{22} v_2^2 \end{aligned}$$

where, in the second line, I used the fact that $A_{12} = A_{21}$. Differentiating with respect to v_1, we find that $\partial S/\partial v_1 = 2A_{11}v_1 + 2A_{12}v_2 = 2\sum_j A_{1j} v_j$, which is the claimed result for $i = 1$. The case $i = 2$ works in the same way.

If there are n variables, then, before differentiating with respect to v_i, it helps to separate out the terms that depend on v_i from those that do not:

$$\begin{aligned} S = \sum_{j,k} A_{jk} v_j v_k &= A_{ii} v_i^2 + \sum_{k \neq i} A_{ik} v_i v_k + \sum_{j \neq i} A_{ji} v_j v_i + \text{terms not involving } v_i \\ &= A_{ii} v_i^2 + 2 \sum_{j \neq i} A_{ij} v_i v_j + \text{terms not involving } v_i. \end{aligned}$$

Here, in passing to the second line, I replaced the dummy index k by j in the sum $\sum_{k \neq i}$, and used the fact that $A_{ji} = A_{ij}$ in the sum $\sum_{j \neq i}$. Differentiating with respect to v_i we find that

$$\frac{\partial S}{\partial v_i} = 2A_{ii} v_i + 2 \sum_{j \neq i} A_{ij} v_j = 2 \sum_j A_{ij} v_j \, .$$

7.46 ⋆⋆ **(a)** A rotation through angle ϵ about the z axis changes the coordinates of particle α thus: $(r_\alpha, \theta_\alpha, \phi_\alpha) \to (r_\alpha, \theta_\alpha, \phi_\alpha + \epsilon)$. Therefore, the invariance of $\mathcal{L}$ when the whole system undergoes this rotation means that

$$\mathcal{L}(r_1, \theta_1, \phi_1 + \epsilon, \cdots, r_N, \theta_N, \phi_N + \epsilon) = \mathcal{L}(r_1, \theta_1, \phi_1, \cdots, r_N, \theta_N, \phi_N).$$

By the definition of partial derivatives, the difference between the two sides of this equation is

$$\text{difference} = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \epsilon = 0 \quad \Longrightarrow \quad \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0. \tag{xvii}$$

(b) Lagrange's equations tell us that $\partial\mathcal{L}/\partial\phi_\alpha = (d/dt)(\partial\mathcal{L}/\partial\dot{\phi}_\alpha) = d\ell_{\alpha z}/dt$. (Recall that $\partial\mathcal{L}/\partial\dot{\phi}_\alpha = \ell_{\alpha z}$, the z component of the angular momentum of particle α.) Therefore the result (xvii) implies that $(d/dt)\sum \ell_{\alpha z} = 0$; that is, the z component of the total angular momentum is constant, $L_z = \sum \ell_{\alpha z} = \text{const}$.

7.48 ⋆⋆ If $F = F(q_1, \cdots, q_n)$, then $dF/dt = \sum_j \dot{q}_j \partial F/\partial q_j$. Therefore, if $\mathcal{L}' = \mathcal{L} + dF/dt$, its derivatives are

$$\frac{\partial \mathcal{L}'}{\partial q_i} = \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j \tag{xviii}$$

and

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}$$

so

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_j \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j\,. \tag{xix}$$

If you compare the two equations (xviii) and (xix), you will see that the two last terms are identical. Thus if $\mathcal{L}$ satisfies Lagrange's equation, so does $\mathcal{L}'$, and vice versa.

7.49 ⋆⋆ **(a)** If $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} = \frac{1}{2}(B_y z - B_z y, B_z x - B_x z, B_x y - B_y x)$, then

$$(\boldsymbol{\nabla}\times\mathbf{A})_x = \partial_y A_z - \partial_z A_y = B_x.$$

(Remember that $\mathbf{B}$ is uniform and constant.) Since the y and z components work the same way, we conclude that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. In polar coordinates, $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B}\times\mathbf{r} = \tfrac{1}{2}B\hat{\mathbf{z}}\times(\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) = \tfrac{1}{2}B\rho\hat{\boldsymbol{\phi}}$$

since $\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}$.

(b) Since there is no electric field, $V = 0$, and since $\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$,

$$\mathcal{L} = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}}\cdot\mathbf{A} = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \tfrac{1}{2}qB\rho^2\dot{\phi}.$$

The three Lagrange equations are

$$m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi}, \qquad \frac{d}{dt}\left(m\rho^2\dot{\phi} + \tfrac{1}{2}qB\rho^2\right) = 0, \qquad \text{and} \qquad m\ddot{z} = 0.$$

(c) In any case, the solution of the z equation is $z = z_o + v_{zo}t$; that is, the particle moves uniformly in the direction of $\mathbf{B}$. If $\rho =$ constant, the ρ equation reduces to $m\dot{\phi}^2 + qB\dot{\phi} = 0$. Therefore, either $\dot{\phi} = 0$ (in which case the particle moves straight along a field line) or $\dot{\phi} = -qB/m$. In this second case, the particle moves clockwise around the z axis (assuming q is positive) at the same time it moves in the z direction with constant velocity; this results in the helical motion described in Section 2.7, with angular velocity equal to the cyclotron freqency $\omega = qB/m$.

7.50 ⋆ The constraint equation is

$$f(x, y) = x + y = \text{const}.$$

The Lagrangian is $\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_2 g y$, and the two modified Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \Longrightarrow \qquad 0 + \lambda = m_1\ddot{x}$$

and

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \Longrightarrow \qquad m_2 g + \lambda = m_2 \ddot{y}.$$

These three equations are easily solved for the three unknowns, $\ddot{x}$, $\ddot{y}$, and λ, to give $\ddot{y} = -\ddot{x} = gm_2/(m_1 + m_2)$, and $\lambda = -gm_1m_2/(m_1 + m_2)$. The constraint force on m_2 (for example) is $F^{\text{cstr}} = \lambda \partial f/\partial y = -gm_1m_2/(m_1 + m_2)$, where the minus sign is because the tension in the string acts upward on m_2, whereas we're measuring y downward. If we wrote down the constraint equation and Newton's second law for the two masses, we would get the same three equations (with λ replaced by minus the tension), so we would naturally get the same solutions.

7.52 ⋆ As the string unwinds, it is clear that $x = R\phi$, so the constraint equation is

$$f = x - R\phi = 0. \tag{xxi}$$

The Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx$ and the two modified Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \Longrightarrow \qquad mg + \lambda = m\ddot{x} \tag{xxii}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} + \lambda \frac{\partial f}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \Longrightarrow \qquad 0 - \lambda R = I\ddot{\phi}. \tag{xxiii}$$

Solving these three equations we find that $\ddot{x} = gm/(m + I/R^2)$ and $\ddot{\phi} = \ddot{x}/R$. If you write down Newton's second law as applied to the mass and the wheel, you should get two equations with exactly the form of Eqs.(xxii) and Eqs.(xxiii) except that λ is replaced by $-F^{\text{t}}$, (minus the tension in the string). Naturally these give the same answer for $\ddot{x}$ and $\ddot{\phi}$. The simplest way to identify λ is to compare the Lagrange equation (xxii) with the Newtonian equation to give $\lambda = -F^{\text{t}}$. Since the constraint function is $f = x - R\phi$, we see that $\lambda \partial f/\partial x = -F^{\text{t}}$, as it should. On the other hand, $\lambda \partial f/\partial \phi = F^{\text{t}}R$, which is the torque on the wheel, as one might have anticipated.
